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From Bi-facial Truth to Bi-facial Proofs

Abstract. In their recent paper *Bi-facial truth: a case for generalized truth values* Zaitsev and Shramko [7] distinguish between an ontological and an epistemic interpretation of classical truth values. By taking the Cartesian product of the two disjoint sets of values thus obtained, they arrive at four generalized truth values and consider two "semi-classical negations" on them. The resulting semantics is used to define three novel logics which are closely related to Belnap's well-known four valued logic. A syntactic characterization of these logics is left for further work.

In this paper, based on our previous work on a functionally complete extension of Belnap's logic, we present a sound and complete tableau calculus for these logics. It crucially exploits the Cartesian nature of the four values, which is reflected in the fact that each proof consists of *two* tableaux. The bi-facial notion of truth of Z&S is thus augmented with a bi-facial notion of proof. We also provide translations between the logics for semi-classical negation and classical logic and show that an argument is valid in a logic for semi-classical negation just in case its translation is valid in classical logic.

Keywords: four-valued logic, bifacial logic, analytic tableaux

1. Introduction

In their recent paper *Bi-facial truth: a case for generalized truth values*, Zaitsev and Shramko [7] (henceforth Z&S) distinguish between an *ontological* (or referential) and *epistemic* (or inferential) interpretation of classical truth values. When the cat is in the garden and Fred thinks it is in the kitchen, 'the cat is in the kitchen' is ontologically false but epistemically true. A little more generally

> If we confine ourselves to just two truth values—truth and falsity, then their referential understanding will be that some sentence *is* (objectively) true or false, and their inferential interpretation means that a sentence *is taken as* (i.e. considered) true (and thus accepted) or false (and thus rejected). [Z&S, p. 1302]

A sentence is either ontologically true or false (and not both) but it is important to stress that the same holds for the epistemic truth values as considered by Z&S. That is, the epistemic truth values are relative to a rational agent who 'never accepts and rejects anything simultaneously, as well as has something to say on any sentence'.

Writing $\{T, F\}$ for the set of ontological truth values and $\{1, 0\}$ for the set of epistemic ones, Z&S let *Cartesian truth values* be elements of the Cartesian product $\{T, F\} \times \{1, 0\}$. Thus, in the situation sketched above, 'the cat is in the kitchen' has the Cartesian truth value $\langle F, 1\rangle$. A combination of the natural order $\leq_t$ on $\{T, F\}$ and the order $\leq_1$ on $\{1, 0\}$, both defined in the expected way, gives rise to a partial order on $\{T, F\} \times \{1, 0\}$ by letting

$$\langle x, y\rangle \leq \langle x', y'\rangle \iff x \leq_t x' \text{ and } y \leq_1 y'$$

The partial order $\leq$ turns $\{T, F\} \times \{1, 0\}$ into $FOUR^{t,1}$, the distributive lattice whose Hasse diagram is depicted below.[*] (Here and below we follow Z&S in writing $T1$ for $\langle T, 1\rangle$ etc. in order to improve readability.)

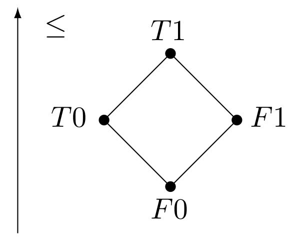

Figure 1. $FOUR^{t,1}$

The order of $FOUR^{t,1}$ gives us a natural definition of entailment for languages whose sentences take values in $\{T, F\} \times \{1, 0\}$: $\alpha \models \beta$ just in case, for every valuation V of our language, the values of α and β are in the $\leq$ ordering. That is:

$$\alpha \models \beta \iff \forall V : V(\alpha) \leq V(\beta) \tag{1}$$

In accordance with (1), Z&S define entailment relations for various propositional languages. All these languages have connectives for conjunction ($\wedge$) and disjunction ($\vee$), which semantically behave as, respectively, meet and join in $FOUR^{t,1}$. Of course, to get an interesting logic one should at least add a negation operator. Several (well-known) options to do so exist. When a *Boolean negation* $\neg_b$ (see Table 1) is added, the entailment relation—induced via (1)—of the resulting language is that of classical logic. When one adds the *De Morgan negation* $\neg_d$ (see Table 1) the entailment relation of the resulting language is that of Belnap's logic, which is axiomatized by a system of first degree entailment as formulated by Anderson and Belnap [1].

[*]Note that $FOUR^{t,1}$ is closely related to Belnap's [3] logical lattice **L4**.

Given the bi-facial picture of truth presented by Z&S, it makes sense to define two less familiar negation operators: *referential negation* ($\neg_t$) and *inferential negation* ($\neg_1$). Referential negation acts as an "ontological truth value swap operator" (leaving the epistemic value unchanged) whereas inferential negation acts as an "epistemic truth value swap operator" (leaving the ontological value unchanged). As both referential and inferential negation act on only one half of a Cartesian truth value, they are called *semi-classical negations* by Z&S.† Table 1 sums up the behaviour of all negations discussed so far.

α	$\neg_t\alpha$	$\neg_1\alpha$	$\neg_b\alpha$	$\neg_d\alpha$
$T1$	$F1$	$T0$	$F0$	$F0$
$T0$	$F0$	$T1$	$F1$	$T0$
$F1$	$T1$	$F0$	$T0$	$F1$
$F0$	$T0$	$F1$	$T1$	$T1$

Table 1. Referential, inferential, Boolean and De Morgan negation.

Both referential and inferential negation ...

> ... turn out to be of complex character, managing to embody some compound content by a sole unitary operator. Namely, for a sentence A its inferential negation $\neg_1 A$ can be informally explicated as "although it is the case that A, an agent denies A", whereas referential negation $\neg_t A$ stands for a composite construction "although it is not the case that A, an agent accepts A". [Z&S, p. 1312]

In accordance with (1), Z&S define entailment relations for propositional languages $\mathcal{L}_t$, $\mathcal{L}_1$ and $\mathcal{L}_{t,1}$, whose logical connectives are contained in $\{\wedge, \vee, \neg_t\}$, $\{\wedge, \vee, \neg_1\}$ and $\{\wedge, \vee, \neg_t, \neg_1\}$ respectively. Although Z&S present lists of theorems and non-theorems for these three *logics of semi-classical negation*, they do not present a syntactic characterization of these logics. In this paper we present, amongst others, a uniform tableau calculus which reflects the "Cartesian product nature" of the four truth values of Z&S, which is, from

†In fact, Z&S give a precise and more general definition of the notion of a semi-classical negation through an algebraic notion of a semi-Boolean complementation, but to restate that definition is not necessary for our purposes.

a formal point of view, the essential feature of their semantics. As Z&S put it

> ... we have developed a method of generalizing classical truth values by taking their Cartesian product. In this way we were able to represent ontological and epistemic aspects of these values within a united semantic framework. [...] as we believe, the very method of Cartesian truth values can open interesting prospects in investigating new logical systems arising from various combinations of logics of different types. [Z&S, p. 1316]

To explain in which sense our calculus reflects the "Cartesian product nature", observe that (1) is equivalent to saying that α entails β just in case, when passing from α to β, both ontological and epistemic truth are preserved. Writing $V_t(\alpha)$ for the ontological component of $V(\alpha) \in \{T, F\} \times \{1, 0\}$ and $V_1(\alpha)$ for its epistemic component, the entailment definition of (1) is thus equivalent to

$$\alpha \models \beta \iff \forall V : V_t(\alpha) = T \Rightarrow V_t(\beta) = T \text{ and } V_1(\alpha) = 1 \Rightarrow V_1(\beta) = 1 \quad (2)$$

We will present a *signed* tableau calculus, whose four signs, T, F, $\mathsf{1}$, and $\mathsf{0}$ wear their interpretation on their sleeves. The syntactic correlate of entailment is then defined in analogy with (2). That is:

$$\alpha \vdash \beta \iff \text{ both } \{\mathsf{T} : \alpha, \mathsf{F} : \beta\} \text{ and } \{\mathsf{1} : \alpha, \mathsf{0} : \beta\} \text{ have closed tableaux} \quad (3)$$

Indeed, (3) states that a proof in our calculus consists of *two* tableaux, one establishing the transmission of ontological truth, the other transmission of epistemic truth. The "bi-facial" tableau calculus of this paper is in an important sense a special case of the "bi-facial" Gentzen calculus—a Gentzen calculus in which each proof consists of two proof trees—for a functionally complete version of Belnap's logic that we presented in Wintein and Muskens [6]. As we explained there, the "bi-facial" nature of our proof system is a natural consequence of the view that Belnap's four values are best understood as combinations of truth values:

> The proof system of the present paper exploits the fact that [Belnap's values True only, False only, Neither and Both] are best thought of as *combinations of truth values*. We choose our four signs to capture the *"underlying" values* of (non-)truth and (non)-falsity and, in doing so we arrive at a proof system that is tailor made for Belnap's logic. [[6], italics added]

Although there are uniform methods by which signed (analytic) tableau calculi for finite valued logics can be obtained (see e.g. Baaz et al. [2]), the calculi that are obtained in particular cases are sometimes unnecessarily complicated and the system for the logics of semi-classical negation that is obtained by the method of [2] is a case in point. The binary connectives, for instance, are provided with tableau rules that have up to four clauses while these clauses themselves may consist of sets of signed statements rather than of single signed statements. Consider, for example, the tableau rule for $T0 : \alpha \wedge \beta$ that is obtained in this manner.

$$\frac{T0 : \alpha \wedge \beta}{T1 : \alpha, T0 : \beta \mid T0 : \alpha, T1 : \beta \mid T0 : \alpha, T0 : \beta}$$

This looks awkward and the awkwardness is explained by the fact the uniform method of [2] is insensitive to the "Cartesian nature" of the four values. In our tableau calculus, which fully recognizes and exploits this nature, a signed tableau rule for a binary connective is either of disjunctive or conjunctive type and always involves exactly two immediate descendants. In this sense, our system closely resembles Smullyan's tableau calculus (cf. Smullyan [5]) for propositional logic.

Z&S provide lists of entailments that, respectively, fail and hold in their logics for semi-classical negation (cf. Proposition 5.1, 5.2 and 5.3). For instance, with respect to the logic of $\mathcal{L}_t$, Z&S remark that $\neg_t(\alpha \vee \beta) \not\models \neg_t\alpha \wedge \neg_t\beta$ but that $\neg_t\alpha \wedge \neg_t\beta \models \neg_t(\alpha \vee \beta)$ is correct. We translate the sentences of $\mathcal{L}_t$, $\mathcal{L}_1$, and $\mathcal{L}_{t,1}$ into a classical propositional language and we then show that an entailment is valid in a logic for semi-classical negation if and only if its translation is valid in classical logic. By doing so, we provide a unified explanation for the occurrence of each entry on the (non-) entailment lists of Z&S.

The rest of the paper is organized as follows. In section 2 we first define our signed tableau calculus for the logics of semi-classical negation and show that these are sound and complete with respect to the (multiple premise, multiple conclusion version of the) semantic entailment relation defined by (1). Then we show that all entailments that are valid in a logic for semi-classical negation are in fact classically valid entailments in disguise. Section 3 briefly discusses a functionally complete extension of $\{\wedge, \vee, \neg_t, \neg_1\}$ and points out that (an extension of) our bi-facial tableau calculus is also sound and complete with respect to the resulting entailment relation. Section 4 concludes.

2. Tableaux and Translations for Semi-classical Negations

In this section, we are concerned with the propositional language $\mathcal{L}_{t,1}$ and its sublanguages $\mathcal{L}_t$ and $\mathcal{L}_1$, as defined in Section 1. An *atomic valuation* v is any function from propositional letters to $\{T, F\} \times \{1, 0\}$. A *valuation* V of $\mathcal{L}_{t,1}$ is the recursive extension of an atomic valuation v to the set of all the sentences of $\mathcal{L}_{t,1}$, in accordance with the truth tables for $\mathcal{L}_{t,1}$'s logical connectives. *Entailment* is defined in accordance with (2), although we think it more natural to work in a setting allowing for finite[‡] sets of premises and conclusions. Thus $\Gamma \models \Delta$ holds just in case in passing from Γ to Δ, both ontological and epistemic truth are preserved.

Definition 1. Let Γ and Δ be sets of $\mathcal{L}_{t,1}$ sentences. Then $\Gamma \models \Delta$ iff $\Gamma \models^o \Delta$ and $\Gamma \models^e \Delta$, where

- $\Gamma \models^o \Delta \iff \forall V : V_t(\alpha) = T$ for all $\alpha \in \Gamma$ implies $V_t(\beta) = T$ for some $\beta \in \Delta$;
- $\Gamma \models^e \Delta \iff \forall V : V_1(\alpha) = 1$ for all $\alpha \in \Gamma$ implies $V_1(\beta) = 1$ for some $\beta \in \Delta$.

We follow Z&S in using $\models_t$ $(\models^o_t, \models^e_t)$ for the restriction of $\models$ $(\models^o, \models^e)$ to sets of $\mathcal{L}_t$ sentences and $\models_1$ $(\models^o_1, \models^e_1)$ for the restriction of $\models$ $(\models^o, \models^e)$ to sets of $\mathcal{L}_1$ sentences.

In this definition V ranges over all $\mathcal{L}_{t,1}$ valuations and V_t and V_1 are defined as in Section 1.

We will capture $\models$ (in terms of $\models^o$ and $\models^e$) via the tableau calculus whose rules are displayed in Table 2.
Names of rules in this calculus will be $\mathsf{T}\wedge$, $\mathsf{0}\neg_t$, etc., with the obvious correspondence between name and rule. The closure conditions of our calculus are as expected.

Definition 2 (Closure conditions). A branch of a tableau is *closed* just in case it contains either $\{\mathsf{T} : \alpha, \mathsf{F} : \alpha\}$ or $\{\mathsf{1} : \alpha, \mathsf{0} : \alpha\}$ for some sentence α. Otherwise the branch is *open*. A tableau is *closed* just in case all its branches are. Otherwise the tableau is *open*.

We define syntactic correlates of $\models^o$, $\models^e$, and $\models$ as follows.

[‡]Working with *finite* sets of sentences is no restriction, as $\models$ is compact. Compactness of $\models$ readily follows from the compactness of the entailment relation of a functionally complete extension of $\mathcal{L}_{t,1}$ (cf. [6]).

$$\frac{\mathsf{T}:\alpha\wedge\beta}{\mathsf{T}:\alpha,\mathsf{T}:\beta}\quad\frac{\mathsf{F}:\alpha\wedge\beta}{\mathsf{F}:\alpha\mid\mathsf{F}:\beta}\qquad\qquad\frac{\mathsf{1}:\alpha\wedge\beta}{\mathsf{1}:\alpha,\mathsf{1}:\beta}\quad\frac{\mathsf{0}:\alpha\wedge\beta}{\mathsf{0}:\alpha\mid\mathsf{0}:\beta}$$

$$\frac{\mathsf{T}:\alpha\vee\beta}{\mathsf{T}:\alpha\mid\mathsf{T}:\beta}\quad\frac{\mathsf{F}:\alpha\vee\beta}{\mathsf{F}:\alpha,\mathsf{F}:\beta}\qquad\qquad\frac{\mathsf{1}:\alpha\vee\beta}{\mathsf{1}:\alpha\mid\mathsf{1}:\beta}\quad\frac{\mathsf{0}:\alpha\vee\beta}{\mathsf{0}:\alpha,\mathsf{0}:\beta}$$

$$\frac{\mathsf{T}:\neg_t\alpha}{\mathsf{F}:\alpha}\quad\frac{\mathsf{F}:\neg_t\alpha}{\mathsf{T}:\alpha}\qquad\qquad\frac{\mathsf{1}:\neg_t\alpha}{\mathsf{1}:\alpha}\quad\frac{\mathsf{0}:\neg_t\alpha}{\mathsf{0}:\alpha}$$

$$\frac{\mathsf{T}:\neg_1\alpha}{\mathsf{T}:\alpha}\quad\frac{\mathsf{F}:\neg_1\alpha}{\mathsf{F}:\alpha}\qquad\qquad\frac{\mathsf{1}:\neg_1\alpha}{\mathsf{0}:\alpha}\quad\frac{\mathsf{0}:\neg_1\alpha}{\mathsf{1}:\alpha}$$

Table 2. Tableau rules for logics of semi-classical negation.

Definition 3. Let Γ and Δ be finite sets of $\mathcal{L}_{t,1}$ sentences. Then $\Gamma \vdash \Delta$ iff $\Gamma \vdash^o \Delta$ and $\Gamma \vdash^e \Delta$, where

- $\Gamma \vdash^o \Delta \iff \{\mathsf{T}:\alpha \mid \alpha \in \Gamma\} \cup \{\mathsf{F}:\beta \mid \beta \in \Delta\}$ has a closed tableau.
- $\Gamma \vdash^e \Delta \iff \{\mathsf{1}:\alpha \mid \alpha \in \Gamma\} \cup \{\mathsf{0}:\beta \mid \beta \in \Delta\}$ has a closed tableau.

The calculus is sound and complete with respect to the logics of semi-classical negation.

Theorem 1. *Let Γ and Δ be finite sets of $\mathcal{L}_{t,1}$ sentences. Then*

1. $\Gamma \models^o \Delta \iff \Gamma \vdash^o \Delta$
2. $\Gamma \models^e \Delta \iff \Gamma \vdash^e \Delta$
3. $\Gamma \models \Delta \iff \Gamma \vdash^o \Delta$ *and* $\Gamma \vdash^e \Delta$

Proof. Similar to the proof of the completeness theorem for an extended logic in the next section. □

From this characterisation of $\models$ we see that in general *two* tableaux need to be made in order to check whether a given sequent is valid, one for the ontological and one for the epistemic part of entailment. Let us illustrate our calculus with an example. As Z&S remark (cf. Proposition 5.2 (1)) we do *not* have $\neg_t(\alpha \vee \beta) \models_t \neg_t\alpha \wedge \neg_t\beta$. Our calculus mimics this judgement syntactically. To see this, consider the relevant ontological and epistemic tableaux:

$$
\begin{array}{c}
\mathsf{T} : \neg_t(\alpha \vee \beta)\\
\mathsf{F} : \neg_t\alpha \wedge \neg_t\beta\\
\mathsf{F} : \alpha \vee \beta\\
\mathsf{F} : \alpha\\
\mathsf{F} : \beta\\
\begin{array}{cc}
\begin{array}{c}\mathsf{F} : \neg_t\alpha\\ \mathsf{T} : \alpha\\ \times\end{array} &
\begin{array}{c}\mathsf{F} : \neg_t\beta\\ \mathsf{T} : \beta\\ \times\end{array}
\end{array}
\end{array}
\qquad
\begin{array}{c}
1 : \neg_t(\alpha \vee \beta)\\
0 : \neg_t\alpha \wedge \neg_t\beta\\
1 : \alpha \vee \beta\\
\begin{array}{cc}
\begin{array}{c} 1 : \alpha\\
\begin{array}{cc}
\begin{array}{c}0 : \neg_t\alpha\\ 0 : \alpha\\ \times\end{array} &
\begin{array}{c}0 : \neg_t\beta\\ 0 : \beta\\ \end{array}
\end{array}\end{array} &
\begin{array}{c} 1 : \beta\\
\begin{array}{cc}
\begin{array}{c}0 : \neg_t\alpha\\ 0 : \alpha\\ \end{array} &
\begin{array}{c}0 : \neg_t\beta\\ 0 : \beta\\ \times\end{array}
\end{array}\end{array}
\end{array}
\end{array}
$$

We see that the ontological tableau on the left is closed whereas the epistemic tableau is not. Thus, our calculus explains the failure of $\neg_t(\alpha \vee \beta) \models_t \neg_t\alpha \wedge \neg_t\beta$: ontological, but not epistemic truth is preserved in passing from $\neg_t(\alpha \vee \beta)$ to $\neg_t\alpha \wedge \neg_t\beta$.

So, whereas in classical logic $\neg(\alpha \vee \beta)$ entails $\neg\alpha \wedge \neg\beta$, its "$\mathcal{L}_t$ counterpart" fails and this failure is explained by the fact that epistemic truth is not preserved. This observation can be generalized. To do so, we let $\mathbf{L}$ be a classical propositional language whose logical connectives are $\{\wedge, \vee, \neg\}$ and we use $\models^{\mathbf{L}}$ to denote classical entailment. For each sentence σ of $\mathbf{L}$, its $\mathcal{L}_t$ *counterpart* σ^t is obtained by replacing each and every occurrence of $\neg$ in σ by $\neg_t$. Also, given a set Γ of $\mathbf{L}$ sentences, we let $\Gamma^t = \{\sigma^t \mid \sigma \in \Gamma\}$. An immediate corollary of the following proposition generalizes the above observation.

Proposition 1. $\Gamma \models^{\mathbf{L}} \Delta \Longleftrightarrow \Gamma^t \models^o_t \Delta^t$.

Proof. ($\Rightarrow$) Consider the ontological tableau rules for referential negation. That is, consider the rules $\mathsf{T}\neg_t$ and $\mathsf{F}\neg_t$. Define the tableau rules $\mathsf{T}\neg$ and $\mathsf{F}\neg$ by replacing, in $\mathsf{T}\neg_t$ and $\mathsf{F}\neg_t$ respectively, $\neg_t$ with $\neg$. The tableau calculus $\mathbf{TL}$, which consists of the rules $\mathsf{T}\wedge$, $\mathsf{F}\wedge$, $\mathsf{T}\vee$, $\mathsf{F}\vee$, $\mathsf{T}\neg$ and $\mathsf{F}\neg$ together with the ontological closure conditions is well-known to be a sound and complete calculus for classical propositional logic (cf. Smullyan [5]). Suppose $\Gamma \models^{\mathbf{L}} \Delta$. Then $\{\mathsf{T} : \alpha \mid \alpha \in \Gamma\} \cup \{\mathsf{F} : \beta \mid \beta \in \Delta\}$ has a closed $\mathbf{TL}$ tableau from which it immediately follows that $\{\mathsf{T} : \alpha^t \mid \alpha \in \Gamma\} \cup \{\mathsf{F} : \beta^t \mid \beta \in \Delta\}$ has a closed tableau in our bi-facial calculus. Hence, $\Gamma^t \models^o_t \Delta^t$.

($\Leftarrow$) Similar to the ($\Rightarrow$) direction. $\square$

Corollary 1. Suppose $\Gamma \models^{\mathbf{L}} \Delta$ and $\Gamma^t \not\models_t \Delta^t$. Then $\Gamma^t \not\models^e_t \Delta^t$.

All the failures of $\mathcal{L}_t$ entailment that are listed by Z&S in their proposition 5.2 are counterparts of classically valid entailments and hence, by Corollary

1, they can all be attributed to failures of transmission of epistemic truth. As inferential negation is dual to referential negation, we also have the following proposition—whose proof is similar to that of Proposition 1—together with an immediate corollary. The notation used in Proposition 2 is as expected: for each $\mathbf{L}$ sentence σ, σ^1 is obtained by replacing each occurrence of $\neg$ in σ with $\neg_1$ and $\Gamma^1 = \{\sigma^1 \mid \sigma \in \Gamma\}$, for each set of $\mathbf{L}$ sentences Γ.

Proposition 2. $\Gamma \models^{\mathbf{L}} \Delta \iff \Gamma^1 \models^e_1 \Delta^1$.

Corollary 2. Suppose $\Gamma \models^{\mathbf{L}} \Delta$ and $\Gamma^1 \not\models_1 \Delta^1$. Then $\Gamma^1 \not\models^o_1 \Delta^1$.

Besides listing failures of $\models_t$ entailments, Z&S also provide a list (cf. proposition 5.1) of classical entailments whose $\mathcal{L}_t$ counterparts *are* $\models_t$ valid. Here is an example: $\neg_t\alpha \wedge \neg_t\beta \models_t \neg_t(\alpha \vee \beta)$. There is a single explanation for the occurrence of each of the 14 valid $\mathcal{L}_t$ entailments that are listed by Z&S.

As $\neg_t\alpha \wedge \neg_t\beta \models_t \neg_t(\alpha \vee \beta)$, it holds in particular that $\neg_t\alpha \wedge \neg_t\beta \models^e_t \neg_t(\alpha \vee \beta)$. Thus, $\{\mathtt{1} : \neg_t\alpha \wedge \neg_t\beta,\ \mathtt{0} : \neg_t(\alpha \vee \beta)\}$ has a closed tableau. But consider the tableau rules $\mathtt{1}\neg_t$ and $\mathtt{0}\neg_t$, i.e. the epistemic rules for referential negation. These rules only "remove" $\neg_t$. So we might just as well remove $\neg_t$ beforehand: as $\neg_t\alpha \wedge \neg_t\beta \vdash^e \neg_t(\alpha \vee \beta)$ and as the $\mathtt{1}\neg_t$ and $\mathtt{0}\neg_t$ rule are trivial, it must also be the case that $\alpha \wedge \beta \vdash^e \alpha \vee \beta$, which is also valid in classical logic.

To explain that this observation holds in full generality, we define for any $\mathcal{L}_t$ sentence σ its $\neg_t$*-free counterpart* $\sigma^\bullet$ by simply removing from σ each occurrence of $\neg_t$. Note that any $\sigma^\bullet$ is an $\mathbf{L}$ sentence as well as an $\mathcal{L}_t$ sentence. For any set of $\mathcal{L}_t$ sentences Γ we let $\Gamma^\bullet = \{\sigma^\bullet \mid \sigma \in \Gamma\}$. We then have the following proposition.

Proposition 3. $\Gamma \models_t \Delta \iff \Gamma^\bullet \models^{\mathbf{L}} \Delta^\bullet$

Proof. ($\Rightarrow$) $\Gamma \models_t \Delta$ implies (per definition) $\Gamma \models^e_t \Delta$ which implies (by completeness) $\Gamma \vdash^e \Delta$. As the epistemic rules for referential negation are trivial, this implies $\Gamma^\bullet \vdash^e \Delta^\bullet$. Now, the epistemic tableau rules pertaining to conjunction and disjunction (together with the epistemic closure conditions) constitute a sound and complete system for classical propositional logic in the connectives $\{\wedge, \vee\}$. Hence $\Gamma^\bullet \models^{\mathbf{L}} \Delta^\bullet$.
($\Leftarrow$) Trivial. $\square$

Of course, we also have a dual proposition. The notation in Proposition 4 is as expected: for any $\mathcal{L}_1$ sentence σ, σ° is obtained from σ by removing all occurrences of $\neg_1$ and, for any set of $\mathcal{L}_1$ sentences Γ, $\Gamma^\circ = \{\sigma^\circ \mid \sigma \in \Gamma\}$.

Proposition 4. $\Gamma \models_1 \Delta \iff \Gamma^\circ \models^{\mathbf{L}} \Delta^\circ$

Proof. Just as the proof of Proposition 3. □

What about the full $\models$ relation on $\mathcal{L}_{t,1}$? For this logic, Z&S also present a list of valid entailments (cf. Proposition 5.3). Again, let us first consider a particular entry of their list, say $\neg_t\alpha \models \neg_1\neg_t\neg_1\alpha$. Now, from $\neg_t\alpha \models \neg_1\neg_t\neg_1\alpha$ it follows, per definition and completeness, that $\neg_t\alpha \vdash^e \neg_1\neg_t\neg_1\alpha$ and $\neg_t\alpha \vdash^o \neg_1\neg_t\neg_1\alpha$. From $\neg_t\alpha \vdash^e \neg_1\neg_t\neg_1\alpha$ and the triviality of $\mathsf{1}\neg_t$ and $\mathsf{0}\neg_t$, it follows that $\alpha \vdash^e \neg_1\neg_1\alpha$. But, modulo a $\neg_1/\neg$ translation, this is just a classically valid entailment, i.e. $\alpha \models^{\mathbf{L}} \neg\neg\alpha$. Dually, from $\neg_t\alpha \vdash^o \neg_1\neg_t\neg_1\alpha$ we can infer that $\neg_t\alpha \vdash^o \neg_t\alpha$, which is, modulo a $\neg_t/\neg$ translation, the classically valid entailment $\neg\alpha \models^{\mathbf{L}} \neg\alpha$.

So, as before, we see that $\models$ valid entailments can be translated into classically valid entailments. In order to make this observation precise, we need some additional notation. As before, $\sigma^\bullet$ will denote the sentence that is obtained by removing all occurrences of $\neg_t$ from σ and σ° will denote the sentence that is obtained by removing all occurrences of $\neg_1$ from σ. Note that $\sigma^\bullet$ is a $\mathcal{L}_1$ sentence and σ° is a $\mathcal{L}_t$ sentence, if σ is an $\mathcal{L}_{t,1}$ sentence. For each σ that is either an $\mathcal{L}_1$ or an $\mathcal{L}_t$ sentence, the $\mathbf{L}$ sentence $\mathcal{C}(\sigma)$ will denote the classical counterpart of σ, obtained by replacing each occurrence of $\neg_t$ or $\neg_1$ in σ with $\neg$. If Γ is a set of sentences, $\Gamma^\bullet$, Γ° and $\mathcal{C}(\Gamma)$ are defined as expected. We then have the following proposition.

Proposition 5. $\Gamma \models \Delta \iff \mathcal{C}(\Gamma^\bullet) \models^{\mathbf{L}} \mathcal{C}(\Delta^\bullet)$ and $\mathcal{C}(\Gamma^\circ) \models^{\mathbf{L}} \mathcal{C}(\Delta^\circ)$.

Proof. The remarks preceding this proposition can be made rigorous along the lines of the proofs of Proposition 1 and Proposition 3. Details are left to the reader. □

3. A functionally complete extension

The tableau rules in Table 2 have the particularity that they do not mix the ontological and the epistemic. A premise signed with T or F will always lead to conclusions signed with T or F and premises signed $\mathsf{1}$ or $\mathsf{0}$ likewise lead to conclusions signed with $\mathsf{1}$ or $\mathsf{0}$. There are, however, operators that cannot be dealt with in this way, as a short glance at the truth table of the negation $\neg_d$ in the introduction will confirm. Given the information that $V_t(\neg_d\alpha) = T$, for example, no conclusion about the value of $V_t(\alpha)$ can be drawn, although it can be concluded that $V_1(\alpha) = 0$.

In fact, tableau rules for $\neg_d$ are easily given and we have done this in the first row of Table 4. Here we have written $\neg_d$ simply as $\neg$, as this negation is the analogue of classical negation in Belnap's logic, and we will continue to

@	$T1$	$T0$	$F1$	$F0$
$T1$	$T1$	$T0$	$T1$	$T0$
$T0$	$T1$	$T0$	$T1$	$T0$
$F1$	$F1$	$F0$	$F1$	$F0$
$F0$	$F1$	$F0$	$F1$	$F0$

$-$	
$T1$	$T1$
$T0$	$F1$
$F1$	$T0$
$F0$	$F0$

Table 3. Truth tables for @ and $-$.

do so. Note that the rules for this operator allow premises signed with T or F to lead to conclusions signed with $\mathsf{1}$ or $\mathsf{0}$ and vice versa. This means that tableaux will no longer either have a purely ontological or a purely epistemic character.

There are more truth-functional operators based on the combinations $T1$, $T0$, $F1$, and $F0$. Here are two of them, @ and $-$, with associated truth tables.
Note that φ @ ψ is ontologically true if and only if φ is ontologically true, but epistemically true iff ψ is. In the literature on bilattices $-$ is sometimes called *conflation*. Two other operators that come natural with Belnap's truth values are $\otimes$ and $\oplus$, which respectively denote meet and join in Belnap's [3] approximation lattice A4. The first of these can be defined by letting $\varphi \otimes \psi$ abbreviate $(\varphi \wedge \psi)$ @ $(\varphi \vee \psi)$ and the second, dually, by letting $\varphi \oplus \psi$ be short for $(\varphi \vee \psi)$ @ $(\varphi \wedge \psi)$. Since Muskens [4] shows that $\{\neg, -, \wedge, \otimes\}$ is a functionally complete set of connectives for Belnap's four-valued logic it follows that $\{\neg, -, \wedge, @\}$ likewise is functionally complete. While it may be hard to provide operators such as @, $\otimes$, $\oplus$, and $-$ with intuitive motivations if the logic is interpreted as a logic of bifacial truth, this nevertheless provides some technical motivation to study them.

The language based on $\{\neg, -, \wedge, @\}$ will be called $\mathcal{L}$ and we now redefine the relation $\models$ to hold between sets of sentences of this language (where V ranges over $\mathcal{L}$ valuations this time and $V(\varphi) = \langle V_t(\varphi), V_1(\varphi)\rangle$ as before).

Definition 4. Let Γ and Δ be sets of $\mathcal{L}$ sentences. Then $\Gamma \models \Delta$ iff $\Gamma \models^o \Delta$ and $\Gamma \models^e \Delta$, where

- $\Gamma \models^o \Delta \iff \forall V : V_t(\alpha) = T$ for all $\alpha \in \Gamma$ implies $V_t(\beta) = T$ for some $\beta \in \Delta$;
- $\Gamma \models^e \Delta \iff \forall V : V_1(\alpha) = 1$ for all $\alpha \in \Gamma$ implies $V_1(\beta) = 1$ for some $\beta \in \Delta$.

$$\frac{\mathsf{T}:\neg\varphi}{\mathsf{0}:\varphi} \qquad \frac{\mathsf{F}:\neg\varphi}{\mathsf{1}:\varphi} \qquad \frac{\mathsf{1}:\neg\varphi}{\mathsf{F}:\varphi} \qquad \frac{\mathsf{0}:\neg\varphi}{\mathsf{T}:\varphi}$$

$$\frac{\mathsf{T}:-\varphi}{\mathsf{1}:\varphi} \qquad \frac{\mathsf{F}:-\varphi}{\mathsf{0}:\varphi} \qquad \frac{\mathsf{1}:-\varphi}{\mathsf{T}:\varphi} \qquad \frac{\mathsf{0}:-\varphi}{\mathsf{F}:\varphi}$$

$$\frac{\mathsf{T}:\varphi\wedge\psi}{\mathsf{T}:\varphi,\ \mathsf{T}:\psi} \qquad \frac{\mathsf{F}:\varphi\wedge\psi}{\mathsf{F}:\varphi \mid \mathsf{F}:\psi} \qquad \frac{\mathsf{1}:\varphi\wedge\psi}{\mathsf{1}:\varphi,\ \mathsf{1}:\psi} \qquad \frac{\mathsf{0}:\varphi\wedge\psi}{\mathsf{0}:\varphi \mid \mathsf{0}:\psi}$$

$$\frac{\mathsf{T}:\varphi\,@\,\psi}{\mathsf{T}:\varphi} \qquad \frac{\mathsf{F}:\varphi\,@\,\psi}{\mathsf{F}:\varphi} \qquad \frac{\mathsf{1}:\varphi\,@\,\psi}{\mathsf{1}:\psi} \qquad \frac{\mathsf{0}:\varphi\,@\,\psi}{\mathsf{0}:\psi}$$

Table 4. Tableau expansion rules for $\neg$, $-$, $\wedge$, and @.

Table 4 gives tableau expansion rules for the connectives $\neg$, $-$, $\wedge$, and @, with closure conditions as before—Definition 2 remains in force. The notions of syntactic derivability are also as before (but with the rules in Table 4 replacing those in Table 2).

DEFINITION 5. Let Γ and Δ be finite sets of $\mathcal{L}$ sentences. Then $\Gamma \vdash \Delta$ iff $\Gamma \vdash^o \Delta$ and $\Gamma \vdash^e \Delta$, where

- $\Gamma \vdash^o \Delta \Longleftrightarrow \{\mathsf{T}:\alpha \mid \alpha \in \Gamma\} \cup \{\mathsf{F}:\beta \mid \beta \in \Delta\}$ has a closed tableau.
- $\Gamma \vdash^e \Delta \Longleftrightarrow \{\mathsf{1}:\alpha \mid \alpha \in \Gamma\} \cup \{\mathsf{0}:\beta \mid \beta \in \Delta\}$ has a closed tableau.

In order to show that the new $\vdash$ indeed characterises the new $\models$, the following two definitions are useful.

DEFINITION 6. Let Θ be a set of signed $\mathcal{L}$ sentences and let V be a $\mathcal{L}$ valuation. We say that V *satisfies* Θ iff the following statements hold.

- $\mathsf{T}:\varphi \in \Theta \Longrightarrow V_t(\varphi) = T$
- $\mathsf{F}:\varphi \in \Theta \Longrightarrow V_t(\varphi) = F$
- $\mathsf{1}:\varphi \in \Theta \Longrightarrow V_1(\varphi) = 1$
- $\mathsf{0}:\varphi \in \Theta \Longrightarrow V_1(\varphi) = 0$

We also say that V *satisfies* ϑ if V satisfies $\{\vartheta\}$.

DEFINITION 7. Let $\mathcal{B}$ be a tableau branch and let $\vartheta \in \mathcal{B}$ be a signed sentence. We say that ϑ is *fulfilled in* $\mathcal{B}$ if ϑ is a signed atomic formula or ϑ instantiates the top formula of a rule in Table 4 while all the corresponding bottom formulas in one of the branches of that rule are also in $\mathcal{B}$. A branch $\mathcal{B}$ is *fulfilled* (or *complete*) if all $\vartheta \in \mathcal{B}$ are fulfilled in $\mathcal{B}$.

So, for example, $\mathsf{F} : \varphi \wedge \psi$ is fulfilled in $\mathcal{B}$ iff either $\mathsf{F} : \varphi \in \mathcal{B}$ or $\mathsf{F} : \psi \in \mathcal{B}$ and similar for the other 15 combinations of signs and connectives. Clearly, every finite tableau can be expanded to a finite tableau in which all branches are fulfilled. This means that if a tableau cannot be expanded to a closed tableau, it can be expanded to a tableau which has a branch that is open and fulfilled.

Note also that, for each instantiation of an expansion rule given in Table 4 (and indeed for each rule in Table 2), it holds that a) if a valuation V satisfies the top formula, V satisfies the bottom formula(s) in one of its branches and b) if a valuation V satisfies all bottom formulas in one branch of the rule, it satisfies the top formula.

These considerations immediately lead to the completeness theorem.

Theorem 2. *Let* Γ *and* Δ *be finite sets of* $\mathcal{L}$ *sentences. Then*

1. $\Gamma \models^o \Delta \Longleftrightarrow \Gamma \vdash^o \Delta$
2. $\Gamma \models^e \Delta \Longleftrightarrow \Gamma \vdash^e \Delta$
3. $\Gamma \models \Delta \Longleftrightarrow \Gamma \vdash^o \Delta$ *and* $\Gamma \vdash^e \Delta$

Proof. In order to prove the left to right direction of 1., suppose $\Gamma \nvdash^o \Delta$. Then $\{\mathsf{T} : \gamma \mid \gamma \in \Gamma\} \cup \{\mathsf{F} : \delta \mid \delta \in \Delta\}$ has a tableau with a branch $\mathcal{B}$ which is open and fulfilled. For all atomic sentences α, define $v_t(\alpha) = T$ if $\mathsf{T} : \alpha \in \mathcal{B}$, $v_t(\alpha) = F$ if $\mathsf{T} : \alpha \notin \mathcal{B}$, $v_1(\alpha) = 1$ if $\mathsf{1} : \alpha \in \mathcal{B}$, and $v_1(\alpha) = 0$ if $\mathsf{1} : \alpha \notin \mathcal{B}$. Let V be the unique valuation such that $V(\alpha) = \langle v_t(\alpha), v_1(\alpha)\rangle$ for all atomic α. Then, since $\mathcal{B}$ is open and by induction on the complexity of formulas, V satisfies $\mathcal{B}$ and hence $\{\mathsf{T} : \gamma \mid \gamma \in \Gamma\} \cup \{\mathsf{F} : \delta \mid \delta \in \Delta\}$, so that $\Gamma \not\models^o \Delta$.

Conversely, suppose that $\Gamma \not\models^o \Delta$, i.e. there is a V such that $V_t(\gamma) = T$, for all $\gamma \in \Gamma$, while $V_t(\delta) = F$, for all $\delta \in \Delta$. Then V satisfies $\{\mathsf{T} : \gamma \mid \gamma \in \Gamma\} \cup \{\mathsf{F} : \delta \mid \delta \in \Delta\}$. Since each rule in the tableau calculus has the property that if a valuation satisfies its top formula it must satisfy the bottom formulas in one of its branches, a tableau starting with $\{\mathsf{T} : \gamma \mid \gamma \in \Gamma\} \cup \{\mathsf{F} : \delta \mid \delta \in \Delta\}$ must have a branch satisfied by V, which must therefore be open, whence $\Gamma \nvdash^o \Delta$.

The proof of 2. is entirely similar and 3. follows from 1. and 2. □

We conclude this section with definitions of the negations given in Z&S's paper (except, of course, their $\neg_d$, which we have taken as a primitive). The reader can easily verify that the tableau expansion rules for $\neg_b$ in Table 5 are derived rules under this definition, as are the rules for $\neg_t$ and $\neg_1$ in Table 2.

$$\frac{\mathsf{T} : \neg_b\alpha}{\mathsf{F} : \alpha} \quad \frac{\mathsf{F} : \neg_b\alpha}{\mathsf{T} : \alpha} \quad \frac{\mathsf{1} : \neg_b\alpha}{\mathsf{0} : \alpha} \quad \frac{\mathsf{0} : \neg_b\alpha}{\mathsf{1} : \alpha}$$

Table 5. Derived tableau expansion rules for $\neg_b$

DEFINITION 8.

$\neg_b\varphi$	abbreviates	$\neg - \varphi$;
$\neg_t\varphi$	abbreviates	$\neg_b\varphi$ @ φ;
$\neg_1\varphi$	abbreviates	φ @ $\neg_b\varphi$.

4. Conclusion

In this paper we have used ideas from [6] in order to provide the logics of semi-classical negation of [7] with analytic tableau calculi. In these calculi proofs in general are based on *two* tableaux. We have also given translations of the logics of semi-classical negation into classical logic which show that their entailment relation mirrors that of the latter. The calculus in [6] for a functional complete extension of Belnap's logic also provides sound and complete rules for the Boolean and De Morgan negations discussed by [7].

Acknowledgements. We would like to thank the anonymous referees for their very useful feedback. Stefan Wintein wants to thank the Netherlands Organisation for Scientific Research (NWO) for funding the project *The Structure of Reality and the Reality of Structure* (project leader: F.A. Muller), in which he is employed. Reinhard Muskens gratefully acknowledges NWO's funding of his project 360-80-050, *Towards Logics that Model Natural Reasoning.*

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