NASSLLI 2016—Multi-Modal Logic

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Part II: Basic Temporal Logic

A Basic Logic of Time

- In this part a basic logic of time will be considered. Temporal logic is a good start for doing modal logics in general.
- (The term 'modal logic' has several senses. Modal logic in the widest sense includes temporal logics; in a narrower sense it only includes logics based on possible worlds. I will use both senses.)
- As I have explained we will introduce temporal logic as a fragment of predicate logic.
- A more customary method is to consider modal logics as logics in their own right. If this course succeeds in whetting your appetite for modal and temporal logics you should certainly study them from this perspective as well.

The Unreality of Time

- In his famous article on *The Unreality of Time* the Scottish idealist philosopher John McTaggart (1866–1925) distinguishes between what he called the A series and the B series of time.
- The A series is related to phrases such as a year ago, two days ago, now, tomorrow, ten years from now, etc. It views time as constantly flowing from the past through the present into the future and gives the present special status in that all times are viewed from its perspective.
- The B series takes an atemporal perspective and views time from an external angle. It is related to such phrases as a year earlier than, two days before, simultaneously with, a day later than, etc.
- The modal account of temporal logic can be viewed as an attempt to model time from an A series perspective.

Temporal Operators

- Prior (1967) wants to formalise temporal logic taking McTaggart's A perspective and considers operators such as

  P — it has at some time been the case that
  F — it will at some time be the case that
  H — it has always been the case that
  G — it will always be the case that

- With these reasonable approximations of the English tenses become possible:

Kjm	John is kissing Mary
$\mathsf{F}Kjm$	John will kiss Mary
$\mathsf{P}Kjm$	John has kissed Mary
$\mathsf{PP}Kjm$	John had kissed Mary
$\mathsf{FP}Kjm$	John will have kissed Mary
$\mathsf{PF}Kjm$	John would kiss Mary
$\mathsf{PFP}Kjm$	John would have kissed Mary

The Language of Propositional Tense Logic

The language of propositional tense logic can now be defined using the following clauses.

1. Any atomic sentence (in a given vocabulary $\mathcal{V}$) is a sentence of propositional tense logic.
2. If φ is a sentence of propositional tense logic then $\neg\varphi$ is too.
3. If φ and ψ are sentences of propositional tense logic, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ are too.
4. If φ is a sentence of propositional tense logic, then $\mathsf{P}\varphi$, $\mathsf{F}\varphi$, $\mathsf{H}\varphi$ and $\mathsf{G}\varphi$ are too.

Translation into Predicate Logic 1

- The language just defined can be provided with its own model theory, but we will follow another path and will give a translation into predicate logic of pairs $t : \varphi$, where t is a term denoting a point of time and φ is a sentence of propositional tense logic.
- $t : \varphi$ can be interpreted as φ holds at t.
- We are shifting back to the B perspective here...
- We will assume that each n-place relation symbol R in the vocabulary of our propositional tense logic is an $n + 1$-place relation symbol R in the vocabulary of the predicate logic we are translating to.
- The first clause of the translation deals with the atomic sentences of the source language:

(1) a. $t : Rc_1 \ldots c_n \triangleq Rc_1 \ldots c_n t$

Translation into Predicate Logic 2

- Here are clauses for the logical connectives (nothing much happens):

(2) a. $t : (\neg\varphi) \triangleq \neg(t : \varphi)$
b. $t : (\varphi \wedge \psi) \triangleq (t : \varphi) \wedge (t : \psi)$
c. $t : (\varphi \vee \psi) \triangleq (t : \varphi) \vee (t : \psi)$
d. $t : (\varphi \rightarrow \psi) \triangleq (t : \varphi) \rightarrow (t : \psi)$
e. $t : (\varphi \leftrightarrow \psi) \triangleq (t : \varphi) \leftrightarrow (t : \psi)$

- And here are clauses for our new operators (with $\prec$ denoting temporal precedence):

(3) a. $t : \mathsf{P}\varphi \triangleq \exists t'(t' \prec t \wedge t' : \varphi)$
b. $t : \mathsf{F}\varphi \triangleq \exists t'(t \prec t' \wedge t' : \varphi)$
c. $t : \mathsf{H}\varphi \triangleq \forall t'(t' \prec t \rightarrow t' : \varphi)$
d. $t : \mathsf{G}\varphi \triangleq \forall t'(t \prec t' \rightarrow t' : \varphi)$

- P translates as 'at some point in the past', G as 'for all points in the future', etc.

Translation into Predicate Logic 3

- The translation rules can now be used to find the translation of any pair $t : \varphi$. Here we find a predicate logical formula equivalent to $t : \mathsf{F}(P \to \mathsf{G}Q)$, for example:
- $$\begin{array}{rl} t : \mathsf{F}(P \to \mathsf{G}Q) & \\ \exists t_1(t \prec t_1 \wedge t_1 : (P \to \mathsf{G}Q)) & \text{using (3b)} \\ \exists t_1(t \prec t_1 \wedge (t_1 : P \to t_1 : \mathsf{G}Q)) & \text{using (2b)} \\ \exists t_1(t \prec t_1 \wedge (t_1 : P \to \forall t_2(t_1 \prec t_2 \to t_2 : Q))) & \text{using (3d)} \\ \exists t_1(t \prec t_1 \wedge (Pt_1 \to \forall t_2(t_1 \prec t_2 \to t_2 : Q))) & \text{using (1a)} \\ \exists t_1(t \prec t_1 \wedge (Pt_1 \to \forall t_2(t_1 \prec t_2 \to Qt_2))) & \text{using (1a)} \end{array}$$
- This gives a semantics. We now know that $\mathsf{F}(P \to \mathsf{G}Q)$ holds at a point t in a model M if and only if $\exists t_1(t \prec t_1 \wedge (Pt_1 \to \forall t_2(t_1 \prec t_2 \to Qt_2)))$ is true in M. Other sentences of propositional tense logic can be treated similarly.

Entailment

- Let's stipulate that $\varphi_1, \ldots, \varphi_n$ entails ψ just if, for all models and all time points $\mathbf{t}$ in those models, ψ is true at time $\mathbf{t}$ if $\varphi_1, \ldots, \varphi_n$ are true at time $\mathbf{t}$.
- This boils down to the statement $\mathbf{t} : \varphi_1, \ldots, \mathbf{t} : \varphi_n \models \mathbf{t} : \psi$ (where $\mathbf{t}$ does not occur in any of $\varphi_1, \ldots, \varphi_n$ and ψ).
- A sentence φ of propositional tense logic is valid if $\models \mathbf{t} : \varphi$.
- Here are some examples of valid sentences (check!):

(4) a. $\mathsf{P}\varphi \leftrightarrow \neg\mathsf{H}\neg\varphi$
b. $\mathsf{F}\varphi \leftrightarrow \neg\mathsf{G}\neg\varphi$
c. $\mathsf{H}(\varphi \rightarrow \psi) \rightarrow (\mathsf{H}\varphi \rightarrow \mathsf{H}\psi)$
d. $\mathsf{G}(\varphi \rightarrow \psi) \rightarrow (\mathsf{G}\varphi \rightarrow \mathsf{G}\psi)$
e. $\varphi \rightarrow \mathsf{HF}\varphi$
f. $\varphi \rightarrow \mathsf{GP}\varphi$

Checking Entailment

- Validities such as the ones on the previous slide can be shown to hold by translation into predicate logic.
- But this is an arduous process. It is also possible to have a direct calculus, with rules such as

$$\dfrac{\mathbf{t} : \neg(\varphi \wedge \psi)}{\mathbf{t} : \neg\varphi \quad \mathbf{t} : \neg\psi}$$

- The following is a justification of this rule. It uses our translational equivalences and the relevant tableau rule for predicate logic.

$$\mathbf{t} : \neg(\varphi \wedge \psi)$$
$$\neg(\mathbf{t} : (\varphi \wedge \psi))$$
$$\neg((\mathbf{t} : \varphi) \wedge (\mathbf{t} : \psi)))$$
$$\neg(\mathbf{t} : \varphi) \quad \neg(\mathbf{t} : \psi)$$
$$\mathbf{t} : \neg\varphi \quad \mathbf{t} : \neg\psi$$

- The next slide will give all propositional rules that can be obtained in this way.

Tableau Rules for Temporal Logic: Propositional Rules

$$\dfrac{\mathbf{t} : \varphi \wedge \psi}{\begin{array}{c}\mathbf{t} : \varphi \\ \mathbf{t} : \psi\end{array}} \qquad \dfrac{\mathbf{t} : \varphi \vee \psi}{\mathbf{t} : \varphi \quad \mid \quad \mathbf{t} : \psi} \qquad \dfrac{\mathbf{t} : \varphi \to \psi}{\mathbf{t} : \neg\varphi \quad \mid \quad \mathbf{t} : \psi} \qquad \dfrac{\mathbf{t} : \varphi \leftrightarrow \psi}{\begin{array}{c}\mathbf{t} : \varphi \\ \mathbf{t} : \psi\end{array} \quad \mid \quad \begin{array}{c}\mathbf{t} : \neg\varphi \\ \mathbf{t} : \neg\psi\end{array}}$$

$$\dfrac{\mathbf{t} : \neg(\varphi \wedge \psi)}{\mathbf{t} : \neg\varphi \quad \mid \quad \mathbf{t} : \neg\psi} \qquad \dfrac{\mathbf{t} : \neg(\varphi \vee \psi)}{\begin{array}{c}\mathbf{t} : \neg\varphi \\ \mathbf{t} : \neg\psi\end{array}} \qquad \dfrac{\mathbf{t} : \neg(\varphi \to \psi)}{\begin{array}{c}\mathbf{t} : \varphi \\ \mathbf{t} : \neg\psi\end{array}} \qquad \dfrac{\mathbf{t} : \neg(\varphi \leftrightarrow \psi)}{\begin{array}{c}\mathbf{t} : \varphi \\ \mathbf{t} : \neg\psi\end{array} \quad \mid \quad \begin{array}{c}\mathbf{t} : \neg\varphi \\ \mathbf{t} : \psi\end{array}}$$

$$\dfrac{\mathbf{t} : \neg\neg\varphi}{\mathbf{t} : \varphi}$$

Just the usual propositional rules with a prefix $\mathbf{t}$...

Rules for the Temporal Operators

- Rules for the temporal operators also fall out of our translation. This mix of translation and tableau:

$$\begin{array}{c} \mathbf{t} : \neg \mathsf{P}\varphi \\ \mathbf{t}_1 \prec \mathbf{t} \\ \neg(\mathbf{t} : \mathsf{P}\varphi) \\ \neg\exists t'(t' \prec \mathbf{t} \wedge t' : \varphi) \\ \forall t'\neg(t' \prec \mathbf{t} \wedge t' : \varphi) \\ \neg(\mathbf{t}_1 \prec \mathbf{t} \wedge \mathbf{t}_1 : \varphi) \\ \begin{array}{cc} \neg\mathbf{t}_1 \prec \mathbf{t} & \neg(\mathbf{t}_1 : \varphi) \\ \times & \mathbf{t}_1 : \neg\varphi \end{array} \end{array}$$

- shows that $\begin{array}{c} \mathbf{t} : \neg \mathsf{P}\varphi \\ \mathbf{t}_1 \prec \mathbf{t} \\ | \\ \mathbf{t}_1 : \neg\varphi \end{array}$ is a derivable rule.
- The next slide lists the rules for temporal operators. They can be justified in a similar way. (Exercise: do this for some of them.)

Tableau Rules for the Temporal Operators

$$\begin{array}{cccc}
\begin{array}{c}\mathbf{t} : \mathsf{P}\varphi \\ | \\ \mathbf{t}_n \prec \mathbf{t} \\ \mathbf{t}_n : \varphi\end{array} &
\begin{array}{c}\mathbf{t} : \mathsf{H}\varphi \\ \mathbf{t}_1 \prec \mathbf{t} \\ | \\ \mathbf{t}_1 : \varphi\end{array} &
\begin{array}{c}\mathbf{t} : \neg\mathsf{P}\varphi \\ \mathbf{t}_1 \prec \mathbf{t} \\ | \\ \mathbf{t}_1 : \neg\varphi\end{array} &
\begin{array}{c}\mathbf{t} : \neg\mathsf{H}\varphi \\ | \\ \mathbf{t}_n \prec \mathbf{t} \\ \mathbf{t}_n : \neg\varphi\end{array}
\end{array}$$

$$\begin{array}{cccc}
\begin{array}{c}\mathbf{t} : \mathsf{F}\varphi \\ | \\ \mathbf{t} \prec \mathbf{t}_n \\ \mathbf{t}_n : \varphi\end{array} &
\begin{array}{c}\mathbf{t} : \mathsf{G}\varphi \\ \mathbf{t} \prec \mathbf{t}_1 \\ | \\ \mathbf{t}_1 : \varphi\end{array} &
\begin{array}{c}\mathbf{t} : \neg\mathsf{F}\varphi \\ \mathbf{t} \prec \mathbf{t}_1 \\ | \\ \mathbf{t}_1 : \neg\varphi\end{array} &
\begin{array}{c}\mathbf{t} : \neg\mathsf{G}\varphi \\ | \\ \mathbf{t} \prec \mathbf{t}_n \\ \mathbf{t}_n : \neg\varphi\end{array}
\end{array}$$

In all cases $\mathbf{t}_n$ must be new.

Note that there are only two patterns of rules—universal and existential.

Equality and Closure Rules

$$\begin{array}{c} \gamma(\mathbf{t}_1) \\ \mathbf{t}_1 = \mathbf{t}_2 \\ | \\ \gamma(\mathbf{t}_2) \end{array} \qquad \begin{array}{c} \gamma(\mathbf{t}_1) \\ \mathbf{t}_2 = \mathbf{t}_1 \\ | \\ \gamma(\mathbf{t}_2) \end{array} \qquad \begin{array}{cc} \diagup & \diagdown \\ \mathbf{t}_1 = \mathbf{t}_2 & \end{array} \qquad \begin{array}{c} \mathbf{t} : \varphi \\ \mathbf{t} : \neg\varphi \\ | \\ \times \end{array}$$

- All constants must be old.
- The third rule is a weakening of the Identification Rule. It ‘tries out’ the possibility that $\mathbf{t}_1 = \mathbf{t}_2$. (let’s call it the Try Out Rule). It comes in handy for finding finite counterexamples and that is its sole motivation.
- Note that identity statements $\mathbf{t}_1 = \mathbf{t}_2$ are not part of the language of temporal logic, just as precedence statements $\mathbf{t}_1 \prec \mathbf{t}_2$ are not. They can occur in tableaus, however.
- The γ in the above range over all tableau entries; the φ over sentences of the temporal logic.

$\models \mathsf{H}(P \to Q) \to (\mathsf{H}P \to \mathsf{H}Q)$

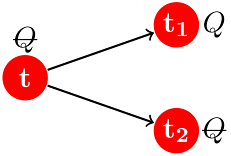
$$\begin{array}{c}
\checkmark\, \mathbf{t} : \neg(\mathsf{H}(P \to Q) \to (\mathsf{H}P \to \mathsf{H}Q)) \\
\mathbf{t} : \mathsf{H}(P \to Q) \\
\checkmark\, \mathbf{t} : \neg(\mathsf{H}P \to \mathsf{H}Q) \\
\mathbf{t} : \mathsf{H}P \\
\checkmark\, \mathbf{t} : \neg\mathsf{H}Q \\
\mathbf{t}_1 \prec \mathbf{t} \\
\mathbf{t}_1 : \neg Q \\
\mathbf{t}_1 : P \\
\checkmark\, \mathbf{t}_1 : P \to Q \\
\begin{array}{cc} \mathbf{t}_1 : \neg P & \mathbf{t}_1 : Q \\ \times & \times \end{array}
\end{array}$$

- An example of a closing tree.
- Items that start with $\mathbf{t} : \mathsf{H}$ are not ticked with a $\checkmark$. They are universal, as are items starting with $\mathbf{t} : \mathsf{G}$, $\mathbf{t} : \neg\mathsf{F}$ and $\mathbf{t} : \neg\mathsf{P}$.

$\not\models \mathsf{F}Q \rightarrow \mathsf{G}Q$

$$
\begin{array}{c}
\checkmark \mathbf{t} : \neg(\mathsf{F}Q \rightarrow \mathsf{G}Q) \\
\checkmark \mathbf{t} : \mathsf{F}Q \\
\checkmark \mathbf{t} : \neg\mathsf{G}Q \\
\mathbf{t} \prec \mathbf{t}_1 \\
\mathbf{t}_1 : Q \\
\mathbf{t} \prec \mathbf{t}_2 \\
\mathbf{t}_2 : \neg Q
\end{array}
$$

- $\mathsf{F}Q \rightarrow \mathsf{G}Q$ obviously should not be valid and it isn't.
- The branch does not close and leads to the following counterexample.

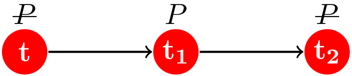


- (The Try Out Rule was not used, but it does not count in deciding whether a branch is finished.)
- Note that $\mathbf{t_1}$ and $\mathbf{t_2}$ are not ordered in either way.

$\not\models \mathsf{G}P \to \mathsf{GG}P$

$$
\begin{array}{c}
\checkmark \mathbf{t} : \neg(\mathsf{G}P \to \mathsf{GG}P) \\
\mathbf{t} : \mathsf{G}P \\
\checkmark \mathbf{t} : \neg\mathsf{GG}P \\
\mathbf{t} \prec \mathbf{t}_1 \\
\checkmark \mathbf{t}_1 : \neg\mathsf{G}P \\
\mathbf{t}_1 : P \\
\mathbf{t}_1 \prec \mathbf{t}_2 \\
\mathbf{t}_2 : \neg P
\end{array}
$$

- This should intuitively be valid, but it isn't! Here is the counterexample we get:



- $\mathbf{t}$ precedes $\mathbf{t}_1$, $\mathbf{t}_1$ precedes $\mathbf{t}_2$, but $\mathbf{t}$ does not precede $\mathbf{t}_2$.
- We need to impose axioms like Transitivity.

Temporal Axioms

- We can add all kinds of axioms, such as our previous set:
A1	$\forall t \, \neg t \prec t$	(irreflexivity)
A2	$\forall t_1 \forall t_2 \forall t_3 ((t_1 \prec t_2 \wedge t_2 \prec t_3) \rightarrow t_1 \prec t_3)$	(transitivity)
A3	$\forall t_1 \forall t_2 (t_1 \prec t_2 \vee t_2 \prec t_1 \vee t_1 = t_2)$	(connectedness)
A4	$\forall t_1 \forall t_2 (t_1 \prec t_2 \rightarrow \exists t_3 (t_1 \prec t_3 \wedge t_3 \prec t_2))$	(density)
A5	$\forall t_1 \exists t_2 \, t_1 \prec t_2$	(no end)
A6	$\forall t_1 \exists t_2 \, t_2 \prec t_1$	(no beginning)
- This embodies the choice that time is a dense linear order without endpoints. Other choices are possible.
- On the next slide we compile these axioms into tableau rules again.

Axioms and Rules

Irreflexivity:

$$\mathbf{t} \prec \mathbf{t}$$

$$\times$$

Transitivity:

$$\mathbf{t}_1 \prec \mathbf{t}_2$$
$$\mathbf{t}_2 \prec \mathbf{t}_3$$

$$\mathbf{t}_1 \prec \mathbf{t}_3$$

Connectedness:

$$\mathbf{t}_1 \prec \mathbf{t}_2 \qquad \mathbf{t}_2 \prec \mathbf{t}_1 \qquad \mathbf{t}_1 = \mathbf{t}_2$$

($\mathbf{t}_1$, $\mathbf{t}_2$ old)

Density:

$$\mathbf{t}_1 \prec \mathbf{t}_2$$

$$\mathbf{t}_1 \prec \mathbf{t}_n$$
$$\mathbf{t}_n \prec \mathbf{t}_2$$

($\mathbf{t}_n$ new)

No End:

$$\mathbf{t} \prec \mathbf{t}_n$$

($\mathbf{t}$ old; $\mathbf{t}_n$ new)

No Beginning:

$$\mathbf{t}_n \prec \mathbf{t}$$

($\mathbf{t}$ old; $\mathbf{t}_n$ new)

$\models_{\mathsf{AX}} \mathsf{G}P \to \mathsf{GG}P$

$\checkmark\mathbf{t} : \neg(\mathsf{G}P \to \mathsf{GG}P)$

$\mathbf{t} : \mathsf{G}P$

$\checkmark\mathbf{t} : \neg\mathsf{GG}P$

$\mathbf{t} \prec \mathbf{t}_1$

$\checkmark\mathbf{t}_1 : \neg\mathsf{G}P$

$\mathbf{t}_1 : P$

$\mathbf{t}_1 \prec \mathbf{t}_2$

$\mathbf{t}_2 : \neg P$

$\mathbf{t} \prec \mathbf{t}_2$

$\mathbf{t}_2 : P$

$\times$

With Transitivity as an axiom $\mathsf{G}P \to \mathsf{GG}P$ becomes valid.

$\models_{AX} \mathsf{F}Q \rightarrow \mathsf{G}(\mathsf{P}Q \vee Q \vee \mathsf{F}Q)$

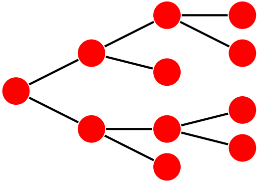
$$\checkmark\, \mathbf{t} : \neg(\mathsf{F}Q \rightarrow \mathsf{G}(\mathsf{P}Q \vee Q \vee \mathsf{F}Q))$$
$$\checkmark\, \mathbf{t} : \mathsf{F}Q$$
$$\checkmark\, \mathbf{t} : \neg\mathsf{G}(\mathsf{P}Q \vee Q \vee \mathsf{F}Q)$$
$$\mathbf{t} \prec \mathbf{t}_1$$
$$\mathbf{t}_1 : Q$$
$$\mathbf{t} \prec \mathbf{t}_2$$
$$\checkmark\, \mathbf{t}_2 : \neg(\mathsf{P}Q \vee Q \vee \mathsf{F}Q)$$
$$\mathbf{t}_2 : \neg\mathsf{P}Q$$
$$\checkmark\, \mathbf{t}_2 : \neg(Q \vee \mathsf{F}Q)$$
$$\mathbf{t}_2 : \neg Q$$
$$\mathbf{t}_2 : \neg\mathsf{F}Q$$

$\mathbf{t}_1 \prec \mathbf{t}_2$	$\mathbf{t}_2 \prec \mathbf{t}_1$	$\mathbf{t}_1 = \mathbf{t}_2$
$\mathbf{t}_1 : \neg Q$	$\mathbf{t}_1 : \neg Q$	$\mathbf{t}_2 : Q$
$\times$	$\times$	$\times$

Crucial use was made of Connectedness.

Branching Time

- The assumption that time is a dense linear order is one among many.
- Many authors are interested in branching time models, where the ordering is allowed to branch in the direction of the future.



- Connectedness is then typically replaced by axioms such as the following:
  $\forall t t_1 t_2((t_1 \prec t \wedge t_2 \prec t) \rightarrow (t_1 \prec t_2 \vee t_2 \prec t_1 \vee t_1 = t_2))$
  $\forall t_1 t_2 \exists t(t \preceq t_1 \wedge t \preceq t_2)$, where $t \preceq t'$ abbreviates $t \prec t' \vee t = t'$

Rules for Branching Time (Dense, No Endpoints)

Irreflexivity, Transitivity, Density, No End, No Beginning +

Tree Condition:

$\mathbf{t}_1 \prec \mathbf{t}$
$\mathbf{t}_2 \prec \mathbf{t}$

$\mathbf{t}_1 \prec \mathbf{t}_2 \quad \mathbf{t}_2 \prec \mathbf{t}_1 \quad \mathbf{t}_1 = \mathbf{t}_2$

Weak Connectedness:

$\mathbf{t}_1 \prec \mathbf{t}_2 \quad \mathbf{t}_2 \prec \mathbf{t}_1 \quad \mathbf{t}_1 = \mathbf{t}_2 \quad \mathbf{t}_n \prec \mathbf{t}_1$
$\mathbf{t}_n \prec \mathbf{t}_2$

($\mathbf{t}_1$, $\mathbf{t}_2$ old; $\mathbf{t}_n$ new)

The Future in Branching Time Models

- The branches in branching time models can be interpreted as histories. Any given point in time can have many possible futures.
- According to the stipulations given thus far, a statement $\mathsf{F}P$ will now be true in $\mathbf{t}$ if P holds in some point $\mathbf{t}'$ in some possible future of $\mathbf{t}$.
- But it seems more reasonable to adopt one of the two following perspectives:
  - a $\mathsf{F}P$ is true in $\mathbf{t}$ iff P is true in some $\mathbf{t}' \succ \mathbf{t}$ in the actual future of $\mathbf{t}$.
  - b $\mathsf{F}P$ is true in $\mathbf{t}$ iff P is true in some $\mathbf{t}' \succ \mathbf{t}$ for each future of $\mathbf{t}$.
- Prior associated the first conception of time with William of Ockham; the second with Charles Saunders Peirce.
- We will return to this point.

Circular Time

- The idea that time is some kind of partial order, linear or branching towards the future, has not always been mainstream.
- The Ancient Greeks, for example, thought of time as cyclical and recurring to the same point over and over again.
- Reynolds (1994) gives a set of axioms for circular time.

Reynolds' Axioms for Cyclical Time

In Reynolds' cyclical time '$t_1 \prec t_2$ is interpreted as 't_1 is before t_2 but not further than half way round the cycle of time'. Here are his axioms.

- $\forall t_1 \forall t_2 (t_1 \prec t_2 \vee t_2 \prec t_1 \vee t_1 = t_2)$
- $\forall t_1 \forall t_2 \neg (t_1 \prec t_2 \wedge t_2 \prec t_1)$
- $\forall t \forall t_1 \forall t_2 \forall t_3 ((t \prec t_1 \wedge t \prec t_2 \wedge t \prec t_3 \wedge t_1 \prec t_2 \wedge t_2 \prec t_3) \rightarrow t_1 \prec t_3)$
- $\forall t \forall t_1 \forall t_2 \forall t_3 ((t_1 \prec t \wedge t_2 \prec t \wedge t_3 \prec t \wedge t_1 \prec t_2 \wedge t_2 \prec t_3) \rightarrow t_1 \prec t_3)$
- $\exists t_1 \exists t_2 \exists t_3 (t_1 \prec t_2 \wedge t_2 \prec t_3 \wedge t_3 \prec t_1)$

The third axiom is called Future Transitivity; the fourth Past Transitivity; and the last Non-transitivity. Easy exercise: compile these into rules (use a closure rule for the asymmetry requirement).

Looking Back and Forward

- We have considered temporal logics as fragments of predicate logic.
- This immediately provided us with tableau rules that are sound (we have not bothered to show they are also complete).
- Adopting sets of axioms often leads to further tableau rules, e.g. for linear time, for branching time, or for cyclical time. There is a tight connection between ontological considerations and proof-theoretic and computational ones here.
- We will now move to Part 3 and see that basic modal logics are very similar to basic temporal logics.