NASSLLI 2016—Multi-Modal Logic

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Part V: A Combination and an Extension

A Combination and an Extension

- In the preceding lectures I have defined one fragment of predicate logic that talks about time and another that talks about other modalities (alethic, deontic, epistemic, doxastic, ...) and conditional modalities. I now want to combine them.
- Up till now the fragments only provided propositional logics. Let's consider possibilities to extend this to quantificational logics.
- Let's start with combining time and modality.

A Word About Sorts/Types

- When we considered our Basic Temporal Logic we used quantification over moments of time to explain it and when we considered modal logics we were quantifying over worlds.
- We now need to combine the two forms of quantification and as soon as we will add quantification over other entities, we must combine three forms.
- We could do this by introducing predicates for times, worlds, and other entities and then restrict our quantifiers to them.
- But this gets rather cumbersome and instead I will assume that the predicate logic we work in is sorted (typed). There will be sorts for times, worlds, and entities and typographical conventions will make clear to what sort an expression belongs.
- t (with or without subscripts or superscripts) will range over times, w over worlds, and x, y, and z over entities, etc.

Combining Time and Modality

We must consider the following.

- How to define the language of our Temporal-Modal Logic.
- How to translate that language into predicate logic.
- What tableau rules will we get as a result.
- How worlds and times are related.

A Decision About Accessibility Relations

- In a context where we have both worlds and times around, what should be done with the accessibility relations?
- Should the precedence relation $\prec$ be made world-dependent in some way? And, should the $R \in \mathcal{R}$ now be made time-dependent?
- We will dogmatically answer the first question with 'no' here (there may be good reasons to do it, but for the moment they do not seem pressing).
- But it seems that the answer to the second question should be 'yes'—since my beliefs change over time, my belief options do. Knowledge, obligations, etc. also change.
- From now on we will assume that the $R \in \mathcal{R}$ are ternary relations between (in that order) a time and two worlds. B and K are now 4-place relations between an entity, a time, and two worlds.

The Language of Propositional Temporal-Modal Logic

The language of propositional temporal-modal (TM) logic will be defined by simply combining the clauses for propositional tense logic and propositional modal logic in the obvious way.

1. Any atomic sentence (in a given vocabulary $\mathcal{V}$) is a sentence of propositional TM logic.
2. If φ is a sentence of propositional TM logic then $\neg\varphi$ is too.
3. If φ and ψ are sentences of propositional TM logic, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ are too.
4. If φ is a sentence of propositional TM logic, then $\mathsf{P}\varphi$, $\mathsf{F}\varphi$, $\mathsf{H}\varphi$ and $\mathsf{G}\varphi$ are too.
5. If φ is a sentence of propositional TM logic and $R \in \mathcal{R}$, then $[R]\varphi$ and $\langle R\rangle\varphi$ are sentences of propositional TM logic.

Translation into Predicate Logic 1

When we provided propositional tense logic with a meaning, we were translating pairs $t : \varphi$, with t denoting a point of time and φ a sentence of propositional tense logic. And when we translated modal logic, we took pairs $w : \varphi$ as input. A simple way to combine this is to work with triples $w, t : \varphi$. Here are clauses for the basic propositional part.

(1) a. $w, t : Rc_1 \dots c_n \triangleq Rc_1 \dots c_n wt$

b. $w, t : (\neg\varphi) \triangleq \neg(w, t : \varphi)$

c. $w, t : (\varphi \wedge \psi) \triangleq (w, t : \varphi) \wedge (w, t : \psi)$

d. $w, t : (\varphi \vee \psi) \triangleq (w, t : \varphi) \vee (w, t : \psi)$

e. $w, t : (\varphi \rightarrow \psi) \triangleq (w, t : \varphi) \rightarrow (w, t : \psi)$

Translation into Predicate Logic 2

And here are clauses for the temporal and modal operators.

(2) a. $w, t : \mathsf{P}\varphi \triangleq \exists t'(t' \prec t \wedge w, t' : \varphi)$

b. $w, t : \mathsf{F}\varphi \triangleq \exists t'(t \prec t' \wedge w, t' : \varphi)$

c. $w, t : \mathsf{H}\varphi \triangleq \forall t'(t' \prec t \rightarrow w, t' : \varphi)$

d. $w, t : \mathsf{G}\varphi \triangleq \forall t'(t \prec t' \rightarrow w, t' : \varphi)$

e. $w, t : [R]\varphi \triangleq \forall w'(Rtww' \rightarrow w', t : \varphi)$, if $R \in \mathcal{R}$

f. $w, t : \langle R \rangle\varphi \triangleq \exists w'(Rtww' \wedge w', t : \varphi)$, if $R \in \mathcal{R}$

The clauses essentially just mirror our previous ones, except that the accessibility relations now talk about worlds and times, not just worlds.

Entailment

- We now define $\varphi_1, \ldots, \varphi_n$ to entail ψ just if, for all models and all world and time points $\mathbf{w}$ and $\mathbf{t}$ in those models, ψ is true at world $\mathbf{w}$ and time $\mathbf{t}$ if $\varphi_1, \ldots, \varphi_n$ are true at $\mathbf{w}$ and $\mathbf{t}$.
- This boils down to the statement $\mathbf{w}, \mathbf{t} : \varphi_1, \ldots, \mathbf{w}, \mathbf{t} : \varphi_n \models \mathbf{w}, \mathbf{t} : \psi$.
- The next slides show the tableau rules obtained from this.

Tableau Rules for TM Logic: Propositional Rules

$$\begin{array}{c} \mathbf{w},\mathbf{t} : \varphi \wedge \psi \\ \hline \mathbf{w},\mathbf{t} : \varphi \\ \mathbf{w},\mathbf{t} : \psi \end{array} \qquad \begin{array}{c} \mathbf{w},\mathbf{t} : \varphi \vee \psi \\ \hline \mathbf{w},\mathbf{t} : \varphi \quad \mid \quad \mathbf{w},\mathbf{t} : \psi \end{array} \qquad \begin{array}{c} \mathbf{w},\mathbf{t} : \varphi \to \psi \\ \hline \mathbf{w},\mathbf{t} : \neg\varphi \quad \mid \quad \mathbf{w},\mathbf{t} : \psi \end{array}$$

$$\begin{array}{c} \mathbf{w},\mathbf{t} : \neg(\varphi \wedge \psi) \\ \hline \mathbf{w},\mathbf{t} : \neg\varphi \quad \mid \quad \mathbf{w},\mathbf{t} : \neg\psi \end{array} \qquad \begin{array}{c} \mathbf{w},\mathbf{t} : \neg(\varphi \vee \psi) \\ \hline \mathbf{w},\mathbf{t} : \neg\varphi \\ \mathbf{w},\mathbf{t} : \neg\psi \end{array} \qquad \begin{array}{c} \mathbf{w},\mathbf{t} : \neg(\varphi \to \psi) \\ \hline \mathbf{w},\mathbf{t} : \varphi \\ \mathbf{w},\mathbf{t} : \neg\psi \end{array}$$

$$\begin{array}{c} \mathbf{w},\mathbf{t} : \neg\neg\varphi \\ \hline \mathbf{w},\mathbf{t} : \varphi \end{array}$$

Just the usual propositional rules with a prefix $\mathbf{w},\mathbf{t}$...

Tableau Rules for the Temporal Operators

$$
\begin{array}{cccc}
\mathbf{w},\mathbf{t} : \mathsf{P}\varphi & \mathbf{w},\mathbf{t} : \mathsf{H}\varphi & \mathbf{w},\mathbf{t} : \neg\mathsf{P}\varphi & \mathbf{w},\mathbf{t} : \neg\mathsf{H}\varphi \\
| & \mathbf{t}_1 \prec \mathbf{t} & \mathbf{t}_1 \prec \mathbf{t} & | \\
\mathbf{t}_n \prec \mathbf{t} & | & | & \mathbf{t}_n \prec \mathbf{t} \\
\mathbf{w},\mathbf{t}_n : \varphi & \mathbf{w},\mathbf{t}_1 : \varphi & \mathbf{w},\mathbf{t}_1 : \neg\varphi & \mathbf{w},\mathbf{t}_n : \neg\varphi
\end{array}
$$

$$
\begin{array}{cccc}
\mathbf{w},\mathbf{t} : \mathsf{F}\varphi & \mathbf{w},\mathbf{t} : \mathsf{G}\varphi & \mathbf{w},\mathbf{t} : \neg\mathsf{F}\varphi & \mathbf{w},\mathbf{t} : \neg\mathsf{G}\varphi \\
| & \mathbf{t} \prec \mathbf{t}_1 & \mathbf{t} \prec \mathbf{t}_1 & | \\
\mathbf{t} \prec \mathbf{t}_n & | & | & \mathbf{t} \prec \mathbf{t}_n \\
\mathbf{w},\mathbf{t}_n : \varphi & \mathbf{w},\mathbf{t}_1 : \varphi & \mathbf{w},\mathbf{t}_1 : \neg\varphi & \mathbf{w},\mathbf{t}_n : \neg\varphi
\end{array}
$$

In all cases $\mathbf{t}_n$ must be new.

Rules for Modal Operators, Equality, Closure

$$\frac{\mathbf{w},\mathbf{t} : [R]\varphi \quad R\mathbf{t}\mathbf{w}\mathbf{w}_1}{\mathbf{w}_1,\mathbf{t} : \varphi} \qquad \frac{\mathbf{w},\mathbf{t} : \neg[R]\varphi}{R\mathbf{t}\mathbf{w}\mathbf{w}_n \quad \mathbf{w}_n,\mathbf{t} : \neg\varphi} \qquad \frac{\mathbf{w},\mathbf{t} : \langle R\rangle\varphi}{R\mathbf{t}\mathbf{w}\mathbf{w}_n \quad \mathbf{w}_n,\mathbf{t} : \varphi} \qquad \frac{\mathbf{w},\mathbf{t} : \neg\langle R\rangle\varphi \quad R\mathbf{t}\mathbf{w}\mathbf{w}_1}{\mathbf{w}_1,\mathbf{t} : \neg\varphi}$$

$$\frac{\gamma(\mathbf{c}_1) \quad \mathbf{c}_1 = \mathbf{c}_2}{\gamma(\mathbf{c}_2)} \qquad \frac{\gamma(\mathbf{c}_1) \quad \mathbf{c}_2 = \mathbf{c}_1}{\gamma(\mathbf{c}_2)} \qquad \frac{}{\mathbf{c}_1 = \mathbf{c}_2 \quad \mid \quad} \qquad \frac{\mathbf{w},\mathbf{t} : \varphi \quad \mathbf{w},\mathbf{t} : \neg\varphi}{\times}$$

The $\mathbf{w}_n$ must be new; all other constants old. R may be replaced by any accessibility relation in the rules for modal operators. The $\mathbf{c}$ can be instantiated as world or as time constants.

Axioms and Rules

- The good news is that we can import our previous axiomatic extensions, provided we take care of the extra argument place that the $R \in \mathcal{R}$ now have.
- In particular, we can again opt to let the temporal precedence relation $\prec$ be ruled by the axioms for dense linear orderings without endpoints, or by the tree axioms for branching time, or by Reynolds axioms for cyclical time, etc.
- Axioms for modal accessibility relations can also be retained. For example, if you want the doxastic alternative relation B_a to be serial, transitive, and euclidean, be my guest. Example rules follow on the next slide.
- (In the previous few slides I have not considered conditional modalities but they can obviously be added.)

Tableau Rules for Belief and Knowledge

Seriality of $\mathsf{B}_a\mathbf{t}$:

$$\frac{\quad}{\mathsf{B}_a\mathbf{t}\mathbf{w}_1\mathbf{w}_n}$$

($\mathbf{t}$, $\mathbf{w}_1$ old, $\mathbf{w}_n$ new)

Transitivity of $\mathsf{B}_a\mathbf{t}$:

$$\frac{\mathsf{B}_a\mathbf{t}\mathbf{w}_1\mathbf{w}_2 \quad \mathsf{B}_a\mathbf{t}\mathbf{w}_2\mathbf{w}_3}{\mathsf{B}_a\mathbf{t}\mathbf{w}_1\mathbf{w}_3}$$

Euclideaness of $\mathsf{B}_a\mathbf{t}$:

$$\frac{\mathsf{B}_a\mathbf{t}\mathbf{w}_1\mathbf{w}_2 \quad \mathsf{B}_a\mathbf{t}\mathbf{w}_1\mathbf{t}\mathbf{w}_3}{\mathsf{B}_a\mathbf{t}\mathbf{w}_2\mathbf{w}_3}$$

Reflexivity of $\mathsf{K}_a\mathbf{t}$:

$$\frac{\quad}{\mathsf{K}_a\mathbf{t}\mathbf{w}\mathbf{w}}$$

($\mathbf{t}$, $\mathbf{w}$ old)

Transitivity of $\mathsf{K}_a\mathbf{t}$:

$$\frac{\mathsf{K}_a\mathbf{t}\mathbf{w}_1\mathbf{w}_2 \quad \mathsf{K}_a\mathbf{t}\mathbf{w}_2\mathbf{w}_3}{\mathsf{K}_a\mathbf{t}\mathbf{w}_1\mathbf{w}_3}$$

Euclideaness of $\mathsf{K}_a\mathbf{t}$:

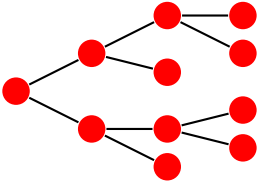
$$\frac{\mathsf{K}_a\mathbf{t}\mathbf{w}_1\mathbf{w}_2 \quad \mathsf{K}_a\mathbf{t}\mathbf{w}_1\mathbf{w}_3}{\mathsf{K}_a\mathbf{t}\mathbf{w}_2\mathbf{w}_3}$$

Worlds and Times

- We now have a system in which worlds and moments of time coexist, but do not interact.
- In this sense the system resembles that of Montague 1973 (but Montague had a necessity operator that quantified over both worlds and times—we haven't).
- We could leave it at that, but there are more interesting possibilities. Aristotle's idea that the future is open (but the past is immutable) can be modelled and we can have a historical necessity operator.

Worlds as Branches

- In branching time logic there is a long tradition of associating worlds ('histories') with branches in the precedence relation.



- There are several ways to do this. One is to simply identify histories (worlds) with branches (subsets that are linearly ordered and maximally so). This leads to a higher order logic.
- But it is also possible to retain moments of time and worlds as first order objects and axiomatise their interaction. Proponents of this 'geometric' view are Van Benthem 1999 and Zanardo 2006.

‘Geometric’ Axioms (Zanardo 2006)

- The geometric approach typically makes use of an ‘incidence’ relation saying that t occurs in w. We will formalise this as $t \star w$.
- The following is from Zanardo 2006. Start with tree axioms (Irreflexivity, Transitivity, Density, No End, No Beginning, the Tree Condition) and add:
  $\forall t \exists w \, t \star w$
  $\forall w \exists t \, t \star w$
  $\forall w \forall t \forall t' ((t \star w \wedge t' \star w) \rightarrow (t \prec t' \vee t' \prec t \vee t = t'))$
  $\forall w \forall t (t \star w \rightarrow \forall t' (t' \prec t \rightarrow t' \star w))$
  $\forall w \forall w' (\forall t (t \star w \rightarrow t \star w') \rightarrow w = w')$
- You will by now have no difficulty to write down tableau rules corresponding to the first four of these axioms. The last one presents a difficulty, but we do not want to adopt it anyway.

Ockhamist and Peircean Conceptions of Time Again

- Our translation $w, t : \mathsf{F}\varphi \triangleq \exists t'(t \prec t' \wedge w, t' : \varphi)$ embodies something that is close to an Ockhamist perspective on the future—something will be true if it is true at a future point in the world of evaluation.
- But on the basis of Zanardo's first four axioms we can now define a ternary relation $\mathcal{H}$ between moments of time and worlds by stipulating that

$$\forall t \forall w \forall w' (\mathcal{H}tww' \leftrightarrow (t \star w \leftrightarrow t \star w'))$$

- We now assume that $\mathcal{H} \in \mathcal{R}$, which makes $[\mathcal{H}]$ a historical necessity operator and $\langle \mathcal{H} \rangle$ express historical possibility. The Peircean view on 'John will kiss Mary' is now formalised by $[\mathcal{H}]\mathsf{F}\, Kjm$—it is settled that he will.

A Simple Alternative I

- I will now sketch a simple alternative to the branching time approach. It is inspired by Kaufmann, Condoravdi & Harizanov 2006.
- There will be a single linear flow of time. We have Irreflexivity, Transitivity, Density, No End, No Beginning, and Connectedness.
- We'll also have the following axioms.
  $\forall t \forall w\, \mathcal{H}tww$
  $\forall t \forall w \forall w' (\mathcal{H}tww' \rightarrow \mathcal{H}tw'w)$
  $\forall t \forall w \forall w' \forall w'' ((\mathcal{H}tww' \wedge \mathcal{H}tw'w'') \rightarrow \mathcal{H}tww'')$
  $\forall t \forall t' \forall w \forall w' (t' \prec t \rightarrow (\mathcal{H}tww' \rightarrow \mathcal{H}t'ww'))$
  $\forall t \forall w \forall w' ((\mathcal{H}tww' \wedge w, t : \varphi) \rightarrow w', t : \varphi)$, if φ is atomic
  $\forall t \forall w \forall w' \forall w'' ((\mathcal{H}tww' \wedge Rtww'') \rightarrow Rtw'w'')$, if $R \in \mathcal{R}$

A Simple Alternative II

- The first three axioms on the previous slide say that the historical alternative relation $\mathcal{H}t$ is an equivalence relation.
- The fourth says that if w and w' are equivalent at some moment in time, they are equivalent at all earlier moments.
- The fifth axiom is a requirement already made by Prior: worlds can only be equivalent at some time if they satisfy the same atomic statements.
- The sixth axiom is similar: worlds that are equivalent at some time stand in the same modal accessibility relations to other worlds at that time.
- I leave a formulation of the tableau rules to you.

Aristotle's Sea-Battle

- There is a famous passage in Aristotle's De Interpretatione where he discusses the problem of future contingents.
- How to interpret the following two statements?
  - Tomorrow there will be a sea-battle
  - Tomorrow there will not be a sea-battle
- There seems to be a conflict between the principle of bivalence and the idea that the future is open, because if one of these sentences is true now this fixes to a certain extent fixes the future.
- And if all future contingents are either true or false, it seems that the future is completely fixed.

The Ockhamist Solution

- If statements are evaluated wrt world-time pairs and if a branching model of time is chosen, as we have essentially done in the above, the Ockhamist and Peircean solutions to Aristotle's puzzle can be explained as follows.
- Ockhamists hold that statements about the future have the form $\mathsf{F}\varphi$. Since $w, t : \mathsf{F}\varphi \triangleq \exists t'(t \prec t' \wedge w, t' : \varphi)$, the sentence *tomorrow there will be a sea-battle* will be true or false and *tomorrow there will not be a sea-battle* will take the opposite value.
- But the future is still open. In fact the indeterminacy of future contingents derives from the indeterminacy of the question in which world we are (in w or in some indistinguishable w' such that $\mathcal{H}tww'$?).

The Peircean Solution

- A Peircean takes the view that statements about the future have the form $[\mathcal{H}]\mathsf{F}\varphi$. This is true in some w and t if $\forall w'(\mathcal{H}tww' \rightarrow \exists t'(t \prec t' \wedge w', t' : \varphi))$.
- So future contingents (contingent wrt historical necessity) are false. Both *tomorrow there will be a sea-battle* and *tomorrow there will not be a sea-battle* are false.
- Both bivalence and the idea of an open future are preserved again.
- It seems to me that the Peircean approach becomes interesting once we assume that context restricts the historical alternatives we are quantifying over.
- *John will send Mary an email tomorrow*—in all historical alternatives we are not ignoring the email is sent.

Adding Quantifiers

- Until now we have only worked with propositional modal fragments of predicate logic. The only form of quantification resided in the modalities, which are guarded quantifiers over worlds and times.
- I will now say a few words about extending the treatment so that we also have quantifiers over individuals in the modal language.

Domains

- If we want to introduce the quantifiers $\exists$ and $\forall$ into the modal language, we will need to make a decision about how $w, t : \exists\varphi$ and $w, t : \forall\varphi$ will be evaluated.
- Will these quantifiers quantify over the full domain of objects in the predicate logical model? It seems that would be inadequate. The sentence *all NASSLLI students are bright* says something about NASSLLI students in this world, not in other worlds.
- Each world, and arguably each world-time pair, must have its own domain of quantification.
- Fortunately, we do not literally need to have new domains in the translating logic. We can have an existence predicate.

Existence Predicate

- We will have an existence predicate E in the underlying predicate logic to restrict quantification. $Exwt$ will render 'x exists in world w at time t'. (E will be ternary in the predicate logic but 1-place in the modal logic, where there will be atoms Ex.)
- You may wonder whether it is necessary to have an argument for a moment of time in E. We tend to think of existence as something eternal (e.g. Aristotle is thought to exist), but there is an alternative conception that takes existence to end after death or destruction ("it is no more").
- An object x that does not exist at w and t is now still possible at w and t if there are w' and t' such that $Exw't'$.

Language and Translation of Quantificational Modal Logic

- We add $\exists$ and $\forall$ to the language in the standard way. Atomic sentences may now contain variables. A term is a variable or a constant.
- E will be a 1-place predicate of the modal language. $\tau_1 = \tau_2$ will be an atomic sentence if τ_1 and τ_2 are terms.
- Note that the terms τ_1 and τ_2 get what is called rigid designation in the translation below—their denotation is world and time independent.

$$w, t : \exists x \varphi \triangleq \exists x (Exwt \wedge w, t : \varphi)$$

$$w, t : \forall x \varphi \triangleq \forall x (Exwt \rightarrow w, t : \varphi)$$

$$w, t : \tau_1 = \tau_2 \triangleq \tau_1 = \tau_2$$

Rules for the Quantifiers

$$\begin{array}{c} \mathbf{w}, \mathbf{t} : \forall x \varphi(x) \\ E\mathbf{c} \\ | \\ \mathbf{w}, \mathbf{t} : \varphi(\mathbf{c}) \end{array} \qquad \begin{array}{c} \mathbf{w}, \mathbf{t} : \neg\forall x \varphi(x) \\ | \\ E\mathbf{c}_n \\ \mathbf{w}, \mathbf{t} : \neg\varphi(\mathbf{c}_n) \end{array} \qquad \begin{array}{c} \mathbf{w}, \mathbf{t} : \exists x \varphi(x) \\ | \\ E\mathbf{c}_n \\ \mathbf{w}, \mathbf{t} : \varphi(\mathbf{c}_n) \end{array} \qquad \begin{array}{c} \mathbf{w}, \mathbf{t} : \neg\exists x \varphi(x) \\ E\mathbf{c} \\ | \\ \mathbf{w}, \mathbf{t} : \neg\varphi(\mathbf{c}) \end{array}$$

The $\mathbf{c}_n$ must be new. All other rules remain the same, taking into account that the identity rules will now also hold for the individual terms that have been introduced.

Looking Back

- A system of quantificational multi-modal logic has now been introduced in which temporal operators and several modal operators in the narrower sense have been combined.
- The system is provided with a tableau system and can be extended and changed in several ways—for example, by changing the axioms regulating the behaviour of accessibility relations or introducing new accessibility relations and providing axioms for them.
- I hope you will play with the system!
- I found it really really enjoyable to teach this class, so thank you for your participation!